On the spherically-symmetric turbulent accretion

V. S. Beskin and K. V. Lezhnin

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4 Two toy models
- A'la Bondi
- A'la Shakura-Sunyaev

Hydrodynamical limit

- axisymmetric ($\partial/\partial\phi = 0$)
- stationary ($\partial/\partial t = 0$)
- ideal (viscosity and heat conduction neglected)

$$n\mathbf{v}_{\mathrm{p}} = \frac{\nabla\Phi \times \mathbf{e}_\varphi}{2\pi r \sin\theta}. \tag{1}$$

- $\nabla \cdot (n\mathbf{v}) = 0$ is satisfied automatically.
- $\mathrm{d}\Phi = n\mathbf{v} \cdot \mathrm{d}\mathbf{S}$
- As $\mathbf{v} \cdot \nabla\Phi = 0$, the velocity vectors $\mathbf{v}$ are located on the surfaces $\Phi(r, \theta) = \mathrm{const}$.

$$
\begin{aligned}
E &= E(\Phi) = \frac{v^2}{2} + w + \varphi_{\rm g}, && (2)\\
L &= L(\Phi) = v_\varphi r \sin\theta, && (3)\\
s &= s(\Phi), && (4)
\end{aligned}
$$

where w is the specific enthalpy, and $\varphi_{\rm g}$ is the gravitational potential.

We consider $s = s(\Phi) = const$. Thus, equation for stream function (no more than Euler equation) looks like (Heyvaerts, 1996):

$$\varpi^2 \nabla_k \left(\frac{1}{\varpi^2 n} \nabla^k \Phi \right) + 4\pi^2 n L \frac{\mathrm{d}L}{\mathrm{d}\Phi} - 4\pi^2 \varpi^2 n \frac{\mathrm{d}E}{\mathrm{d}\Phi} = 0, \qquad (5)$$

where $\varpi = r \sin\theta$.
For spherically symmetric case, i.e., for $E(\Phi) = \mathrm{const}$, $L(\Phi) = 0$, the solution is: it has the solution

$$\Phi = \Phi_0(1 - \cos\ \theta). \qquad (6)$$

Angular operator (appears in (5))

$$\hat{\mathcal{L}}_\theta = \sin\theta \frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right), \tag{7}$$

its eigenfunctions:

$$Q_0 = 1 - \cos\ \theta, \tag{8}$$

$$Q_1 = \sin^2\theta, \tag{9}$$

$$Q_2 = \sin^2\theta\ \cos\ \theta, \tag{10}$$

$$\ldots$$

$$Q_m = \frac{2^m m!(m-1)!}{(2m)!}\sin^2\theta\ \mathcal{P}'_m(\cos\ \theta), \tag{11}$$

and eigenvalues

$$q_m = -m(m+1). \tag{12}$$

$$\Phi = \Phi_0[1 - \cos\ \theta + \varepsilon^2 f(r,\theta)], \tag{13}$$

where ε is the small parameter $\ll 1$. Equation (5) can be linearised, while the equation for the perturbation function $f(r,\theta)$ is written as:

$$\begin{aligned} -\varepsilon^2 D\frac{\partial^2 f}{\partial r^2} - \frac{\varepsilon^2}{r^2}(D+1)\sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial f}{\partial\theta}\right) + \varepsilon^2 N_r\frac{\partial f}{\partial r} = \\ = -\frac{4\pi^2 n^2 r^2}{\Phi_0^2}\sin\theta(D+1)\frac{\mathrm{d}E}{\mathrm{d}\theta} \\ +\frac{4\pi^2 n^2}{\Phi_0^2}(D+1)\frac{L}{\sin\theta}\frac{dL}{d\theta} - \frac{4\pi^2 n^2}{\Phi_0^2}\frac{\cos\theta}{\sin^2\theta}L^2. \end{aligned} \tag{14}$$

Here $D = -1 + c_{\mathrm{s}}^2/v^2$, and $N_r = 2/r - 4\pi^2 n^2 r^2 GM/\Phi_0^2$.

Seeking for solution in the form

$$f(r,\theta) = \sum_{m=0}^{\infty} g_m(r) Q_m(\theta). \tag{15}$$

Introducing dimensionless variables

$$x = \frac{r}{r_*},\ u = \frac{n}{n_*},\ l = \frac{c_{\mathrm{s}}^2}{c_*^2}, \tag{16}$$

where the $*$-values correspond to the sonic surface. Thus, equation for radial functions $g_m(r)$

$$\begin{aligned}(1 - x^4 l u^2) g_m'' + 2\left(\frac{1}{x} - x^2 u^2\right) g_m' + m(m+1) x^2 l u^2 g_m = \\ = k_m \frac{R^2}{r_*^2} x^4 l u^4 - \lambda_m \frac{R^2}{r_*^2} u^2 - \sigma_m x^6 l u^4.\end{aligned} \tag{17}$$

$$\sin\theta \frac{\mathrm{d}E}{\mathrm{d}\theta} = \varepsilon^2 c_*^2 \sum_{m=0}^{\infty} \sigma_m Q_m(\theta), \tag{18}$$

$$\frac{\cos\theta}{\sin^2\theta} L^2 = \varepsilon^2 c_*^2 r_*^2 \sum_{m=0}^{\infty} \lambda_m Q_m(\theta), \tag{19}$$

$$\frac{L}{\sin\theta} \frac{\mathrm{d}L}{\mathrm{d}\theta} = \varepsilon^2 c_*^2 r_*^2 \sum_{m=0}^{\infty} k_m Q_m(\theta). \tag{20}$$

$l(x)$ and $u(x)$: $P(n,s) = A(s) n^{\Gamma-1} \to l = u^{\Gamma-1}$.

$$\frac{\mathrm{d}u}{\mathrm{d}x} = -2\frac{u}{x}\frac{(1 - x^3 u^2)}{(1 - x^4 l u^2)} \tag{21}$$

$$u(x)|_{x=1} = 1, \tag{22}$$

$$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=1} = -\frac{4 - \sqrt{10 - 6\Gamma}}{\Gamma + 1}. \tag{23}$$

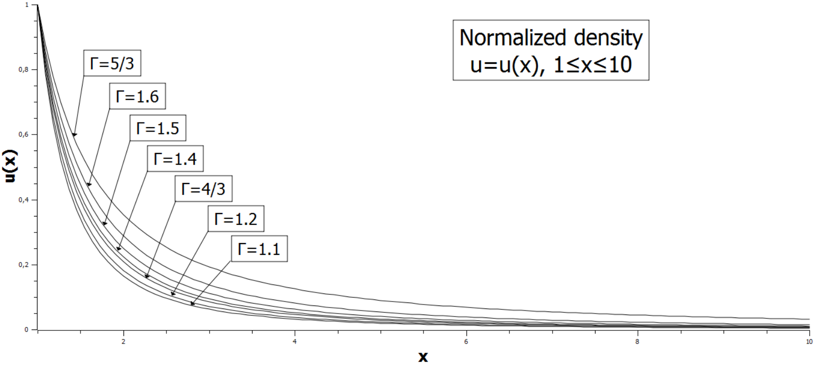

Figure: $u(x)$ in region $1 \leq x \leq R/r_* = 10$, Γ from 1.1 to 5/3

$v_\theta|_{r=R} = 0$ and absense of singularity on sonic surface give

$$\varepsilon^2 g_m(1) = \frac{(2m)!}{2^m(m+1)m!}\left[\frac{(\delta E_{\mathrm{n}})_m}{c_*^2} - \frac{(L_n^2/\sin^2\theta)_m}{2c_*^2 r_*^2}\right], \qquad (24)$$

$$g_m'(R/r_*) = 0, \qquad (25)$$

where $(\ldots)_m$ stands for the expansion in terms of the Legendre polynomials.

- quasi-spherical accretion
- small and axisymmetric perturbation ($\epsilon \ll 1$ and $\delta\theta_{\mathrm{curl}} \ll 1$)
- the goal is to find $f(r, \theta)$ and determine v_θ and v_ϕ velocity components.

$$\Omega(\theta) = \Omega_0 \exp[-\alpha^2(1 + \cos\ \theta)]. \tag{26}$$

Ω_0 and $\alpha \simeq 10$ are free parameters of our model. Disturbances of energy and momentum integrals

$$\delta E_{\rm n}(\theta) = \frac{\Omega_0^2 \exp[-2\alpha^2(1 + \cos\ \theta)] R^2 \sin^2\theta}{2}, \tag{27}$$

$$\delta L_{\rm n}(\theta) = \Omega_0 \exp[-\alpha^2(1 + \cos\ \theta)] R^2 \sin^2\theta. \tag{28}$$

It gives $\varepsilon = \Omega_0 R / c_*$.

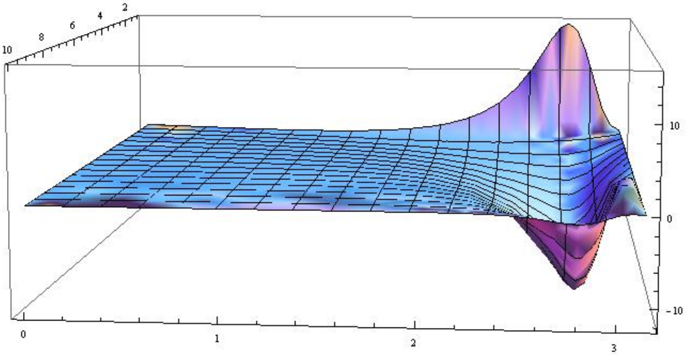

Figure: $f(r,\theta)$ in region $1 \leq x \leq R/r_* = 10$, $0 \leq \theta \leq \pi$, $\Gamma = 4/3$

$x \ll 1$:

$$g_m'' + \frac{3}{2x} g_m' + \frac{m(m+1)}{2^{\Gamma+1}} x^{-(3\Gamma+1)/2} g_m + \lambda_m \frac{R^2}{r_*^2} \frac{1}{4x^3} = O(x^{-3}). \quad (29)$$

Introducing new function $y_m = g_m / (\lambda_m R^2 / r_*^2)$, Eqn. (29) can be rewritten as

$$y_m'' + \frac{3}{2x} y_m' + \frac{m(m+1)}{2^{\Gamma+1}} x^{-(3\Gamma+1)/2} y_m + \frac{1}{4x^3} = O(x^{-3}). \quad (30)$$

This equation has an universal solution independent of the boundary conditions on the outer boundary $r = R$

$$y(x) = -\frac{8}{x}. \quad (31)$$

$$\frac{v_\theta}{v_\varphi} = 2\sqrt{2}\,\varepsilon\pi\left(\frac{r_*}{R}\right)^{7/2} p(r,\theta), \tag{32}$$

where

$$p(r,\theta) = \frac{\sum_{m=0}^{\infty} g_m'(r) Q_m(\theta)}{u(r)\sin^2\theta\exp[-\alpha^2(1+\cos\ \theta)]}. \tag{33}$$

In our calculation, $r_*/R = 0.1$, $|p(r,\theta)| < 20$.
$|f(r,\theta)| < 20 \rightarrow \varepsilon = 10^{-3}$.

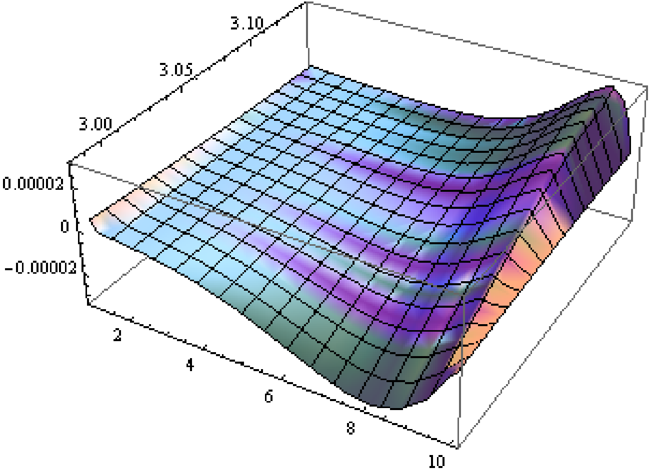

Figure: $|v_\theta / v_\varphi|$ ratio.

$v_\theta \ll v_\varphi$ around the curl, so we neglect all terms in Navier-Stokes equations that contain v_θ.

$$v_r \frac{\partial v_\theta}{\partial r} + v_\theta \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} - \frac{v_\varphi^2}{r} \cot\theta = -\frac{1}{r} \frac{\partial P / \partial \theta}{\rho}. \tag{34}$$

$$-\frac{v_\varphi^2}{r} \cot\theta = -\frac{1}{r} \frac{\partial P / \partial \theta}{\rho}. \tag{35}$$

Expanding near the axis $\theta \ll 1$

$$v_\varphi = \frac{L}{r \sin\theta} = \frac{K\theta}{r}, \tag{36}$$

where $K = \text{const}$,

$$\rho \frac{K^2 \theta}{r^2} = \frac{\partial P}{\partial \theta}, \tag{37}$$

which gives for the pressure P

$$P \sim \frac{\rho K^2}{\alpha^2 r^2}, \tag{38}$$

where α^{-1} is an approximate size of a curl.

$$\begin{aligned}\nabla(n\mathbf{v}) &= 0, &(39)\\ (\mathbf{v}\cdot\nabla)\mathbf{v} &= -\frac{\nabla P}{\rho} - \nabla\varphi_{\mathrm{g}}, &(40)\\ (\mathbf{v}\cdot\nabla)s &= 0. &(41)\end{aligned}$$

$+$ polytropic EOS $P = P(n,s) = k(s)n^{\Gamma}$

$$\begin{aligned}c_{\mathrm{s}}^2 &= \frac{\Gamma k}{m_p} n^{\Gamma-1}, &(42)\\ w &= \frac{c_{\mathrm{s}}^2}{\Gamma-1}, &(43)\\ T &= \frac{m_p}{\Gamma} c_{\mathrm{s}}^2. &(44)\end{aligned}$$

Assuming that $v_r \gg v_\varphi \gg v_\theta$,

$$v_\varphi(r,\theta) = \Omega(\theta)\frac{R^2}{r}\sin\theta. \tag{45}$$

Here $\Omega(\theta)$ is a smooth function of θ that can be approximately described as:

$$\Omega(\theta) \approx \begin{cases} \Omega_0, & \pi - \alpha^{-1} < \theta < \pi, \\ 0 & 0 < \theta < \pi - \alpha^{-1}. \end{cases} \tag{46}$$

$$E(\theta) = \frac{v_r^2(r)}{2} + \omega(r) + \varphi_{\mathrm{g}}(r) + \frac{L^2(\theta)}{2r^2\sin^2\theta}, \tag{47}$$

$$L(\theta) = v_\varphi r\sin\theta = \Omega(\theta) r^2 \sin^2\theta. \tag{48}$$

Averaging now these integrals in θ and introducing a new value

$$L_{\mathrm{av}}^2 \equiv \langle \frac{L^2(\theta)}{\sin^2\theta} \rangle \tag{49}$$

we can indite a new equation for averaged energy integral

$$E_{\mathrm{av}} \equiv \langle E_{\mathrm{n}}(\theta) \rangle = \frac{v_r^2(r)}{2} + \omega(r) + \underbrace{\varphi_{\mathrm{g}}(r) + \frac{L_{\mathrm{av}}^2}{2r^2}}_{\varphi_{\mathrm{eff}}(r)}, \tag{50}$$

Using the total particle flux

$$\Phi = 4\pi r^2 n(r) v_r(r) = const, \tag{51}$$

and the entropy s, one can rewrite the energy integral (50) as

$$E_{\mathrm{av}} = \frac{\Phi^2}{32\pi^2 n^2 r^4} + \frac{\Gamma k(s)}{\Gamma - 1} \frac{n^{\Gamma-1}}{m_p} - \frac{GM}{r} + \frac{L_{\mathrm{av}}^2}{2r^2}. \tag{52}$$

It gives the following expression for the logarithmic r-derivative of the number density

$$\eta_1 = \frac{r}{n} \frac{\mathrm{d}n}{\mathrm{d}r} = \frac{2 - \dfrac{GM}{v_r^2 r} + \dfrac{L_{\mathrm{av}}^2}{v_r^2 r^2}}{-1 + \dfrac{c_{\mathrm{s}}^2}{v_r^2}}. \tag{53}$$

As for Bondi accretion, this derivative has a singularity on the sonic surface $v_r = c_{\mathrm{s}} = c_*$.

This implies that for smooth transition through the sonic surface $r = r_*$, the additional condition is to be satisfied:

$$2 - \frac{GM}{c_*^2 r_*} + \frac{L_{\rm av}^2}{c_*^2 r_*^2} = 0. \tag{54}$$

Solving this equation in terms of r_* and assuming that $L_{\rm av} c_*/(GM) \ll 1$, we find

$$r_* = \frac{GM}{2c_*^2}\left(1 - \frac{4L_{\rm av}^2 c_*^2}{G^2 M^2}\right), \tag{55}$$

$$c_* = \sqrt{\frac{2}{5-3\Gamma}}\,c_\infty \left(1 + \frac{12(\Gamma-1)}{(5-3\Gamma)^2}\,\frac{L_{\rm av}^2 c_\infty^2}{G^2 M^2}\right), \tag{56}$$

where c_∞ is evaluated from

$$E_{\rm av} = \frac{c_\infty^2}{\Gamma - 1}. \tag{57}$$

$$\frac{r_*}{r_{*\mathrm{B}}} = 1 - \left[\frac{16}{(5-3\Gamma)^2}\,\frac{L_{\mathrm{av}}^2 c_\infty^2}{G^2 M^2}\right] \quad (58)$$

$$\frac{c_*}{c_{*\mathrm{B}}} = 1 + \left[\frac{12(\Gamma-1)}{(5-3\Gamma)^2}\,\frac{L_{\mathrm{av}}^2 c_\infty^2}{G^2 M^2}\right], \quad (59)$$

where $c_{*\mathrm{B}}$ and $r_{*\mathrm{B}}$ correspond to classical Bondi accretion. As we see, the nonzero angular momentum effectively decreases the gravitational force.

- axisymmetric ($\partial/\partial\phi = 0$)
- stationary ($\partial/\partial t = 0$)
- viscous flow ($\eta = \rho\nu = const$)

Again, $v_r \gg v_\varphi \gg v_\theta$ gives (ϕ-component of Euler eqn.)

$$v_r \frac{\partial v_\varphi}{\partial r} + \frac{v_r v_\varphi}{r} = \nu \left(\nabla^2 v_\varphi - \frac{v_\varphi}{r^2 \sin^2 \theta} \right). \tag{60}$$

$$v_\varphi = \Omega(r, \theta) r \sin \theta, \tag{61}$$

where we will use the following form for the angular velocity $\Omega(r, \theta)$:

$$\Omega(r, \theta) = \Omega_0(r) \exp \left[-\frac{\theta^2}{2\delta(r)} \right]. \tag{62}$$

Here $\Omega_0 = \Omega_0(r)$ is an amplitude, and $\delta = \delta(r)$ is an effective angular width of an individual curl.

Substituting now v_φ into Eqn. (60), we obtain

$$\dot{M}\frac{\mathrm{d}}{\mathrm{d}r}(\Omega r^2 \sin\theta) = \frac{4\pi r^2 \eta}{\sin^2\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\sin^3\theta \frac{\mathrm{d}\Omega}{\mathrm{d}\theta}\right], \tag{63}$$

where

$$\dot{M} = 4\pi r^2 \rho v_r \tag{64}$$

is the accretion rate remaining constant in stationary flow. Using now Eqn. (63), one can show that the total angular momentum of an individual vortex conserves ($\mathrm{d}L/\mathrm{d}r = 0$). Indeed,

$$\mathrm{d}L = \rho\Omega r^2 \sin^2\theta \mathrm{d}\phi \sin\theta \mathrm{d}\theta r^2 \mathrm{d}r \tag{65}$$

can be rewritten as a full θ-derivative.

(62) → (63) and expanding in θ

$$\frac{\dot{M}}{2\pi\eta} r\Omega_0(r) + \frac{4r^2\Omega_0(r)}{\delta(r)} + \frac{\dot{M}}{4\pi\eta} r^2\Omega_0'(r) = 0, \tag{66}$$

$$-18r^2\Omega_0(r) - \frac{3\dot{M}}{2\pi\eta} r\delta(r)\Omega_0(r) - 10r^2\delta(r)\Omega_0(r) -$$
$$\frac{\dot{M}}{2\pi\eta} r\delta^2(r)\Omega_0(r) + \frac{3\dot{M}}{4\pi\eta} r^2\delta'(r)\Omega_0(r) - \frac{\dot{M}}{4\pi\eta} r^2\delta^2(r)\Omega_0'(r) = 0, \tag{67}$$

that has simple solutions

$$\delta(r) = \delta_0 + \frac{8\pi\eta(r_0 - r)}{\dot{M}}, \tag{68}$$

$$\Omega_0(r) = \Omega_0(\frac{r_0}{r})^2 \left[\frac{8\pi\eta(r_0 - r)}{\dot{M}\delta_0} + 1\right]^{-2}. \tag{69}$$

$$\frac{8\pi\eta r_0}{\dot{M}} \ll 1. \tag{70}$$

Introducing Reynolds number as

$$Re = \frac{\rho v l}{\eta}, \tag{71}$$

where ρ, v and l are characteristic values of a flow and η is a dynamical viscosity, we can transform it using expression (64) and get

$$Re = \frac{\dot{M}}{4\pi r_0 \eta} \gg 1. \tag{72}$$

$$\Omega(r,\theta) = \Omega_0(r)\exp(-\frac{\theta^2}{2\delta^2})(1-\frac{\alpha^2}{2\delta^2}\theta^2), \tag{73}$$

where parameter α is chosen from the condition on total angular momentum

$$\int_{|\vec{r}|\le R} dL = 0, \tag{74}$$

which is equivalent to

$$\int_0^\pi d\theta \sin^3\theta\, \Omega(r,\theta) = 0. \tag{75}$$

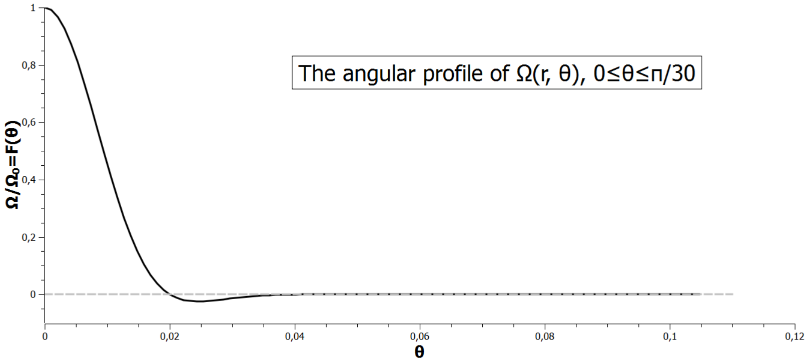

Figure: Ratio $\Omega/\Omega_0 = F(\theta)$, $0 \leq \theta \leq \pi/30$

$$\Omega_0(r) = \Omega_0 \left(\frac{r_0}{r}\right)^2 \exp\left[\frac{16\pi\eta}{\dot{M}}\frac{(1+\alpha^2)}{\delta^2}(r_0 - r)\right]. \tag{76}$$

According to classical prediction of Shakura and Sunyaev, we set kinematical viscosity as

$$\nu = \alpha_{\rm ss} r_{\rm curl} v_{\rm curl} \approx \alpha_{\rm ss} r_0^2 \delta^2 \Omega_0. \tag{77}$$