On the semianalytical two-body regularization in N-body simulations

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Abstract

A two-body regularization for N-body problem based on perturbation theory for Keplerian problem is discussed. We provide analytical estimations of accuracy and conduct N-body experiments in order to compare it with state-of-the-art Hermite integrator. It is shown that this regularization keeps some features that allow to overcome KS-regularization in some particular cases.

The solution technique

The direct solution of the problem under consideration by classical methods runs into great computational difficulties. The first simplification for solving this problem is the individual choice of an integration step for each body (Dehnen and Read 2011). This method makes sense when the stellar system being modeled has a strong density inhomogeneity. In this solution, the step is chosen according to S.J. Aarseths empirical formula

$$\Delta t_i = \sqrt{\eta \frac{a_i \ddot{a}_i + \dot{a}_i^2}{\dot{a}_i \dddot{a}_i + \ddot{a}_i^2}}, \tag{1}$$

where a_i is the magnitude of the acceleration for the i-th body, the dot denotes a time derivative, and η is a dimensionless parameter usually chosen to be ≈ 0.02 (see Dehnen and Read 2011).

Consider the Hamiltonian of a system of point particles interacting gravitationally according to Newton:

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} - \frac{G}{2} \sum_{i=1, j=1, i \neq j}^{N} \frac{m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|}. \tag{2}$$

Below, we will consider the specific pth body and as- sume that the body nearest to it has number k. Let us separate out the part of the Hamiltonian containing the singularity and call it H_0 :

$$H_0 = \frac{p_p^2}{2m_p} + \frac{p_k^2}{2m_k} - \frac{G m_p m_k}{|\mathbf{r}_p - \mathbf{r}_k|}. \tag{3}$$

Let us now assume that the pair formed by the kth and pth bodies is sufficiently close (the tidal forces from the interaction with other bodies are much weaker than the interaction force between the two chosen bodies). The exact trajectories of these two bodies can then be considered as an unperturbed motion, while we will take into account the influence of the surrounding bodies as a small correction. In this case, the integration step will be determined by the perturbation $V = H - H_0$ and can be chosen to be considerably larger than in the standard technique of series expansion. **If this method is compared with the KS regularization, then the main advantage of the approach under consideration is that the integration step of a binary system for $V \to 0$ tends to infinity (it is determined by the slow subsystem), which is not the case when the KS regularization is used (Kustaanheimo, Stiefel, 1965).** Let us now turn to a detailed description of the algorithm. The implemented code allows the collisional N -body problem to be solved; for more rigorous consideration of the algorithm, see (Lezhnin, Chernyagin, 2014).

Numerical testing for three-body problem

Let us make an experimental comparison of the method of direct calculation and the method of integration in action variables for the three-body problem as an example. The initial data are specified in the left column of the code window, the timers in internal units of measurement (Earth years) are located below (see Fig. 1). The trajectories in the left part of the figure were completely calculated by the direct method, while the segments of the trajectories marked by different shades of gray on the right mark the bodies whose positions were calculated by the method described in this paper. **Note that the trajectories not only coincided with a good accuracy (the relative error is $\approx 10^{-6}$), but also the calculation by the proposed method took approximately a factor of 10 less time on the same PC.**

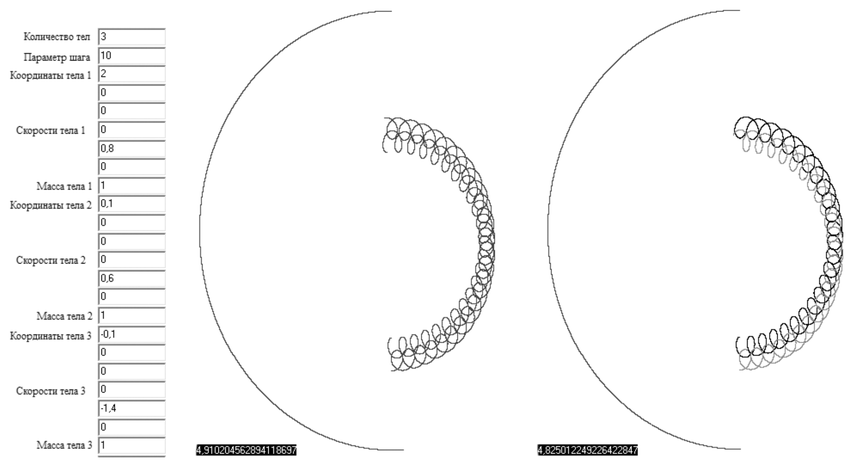

Figure 1: Comparison of orbits derived by straightforward procedure and regularized one

Accuracy testing for N-body problem

In order to check the level of precision that can be achieved by the code described above, we conduct series of N-body experiments and compare the state-of-the-art Hermite code with our version. The algorithm of the verification is the following – we generate an initial snapshot with coordinates and velocities taken randomly in some range. All particles have masses equal to one solar mass. Using these initial conditions, we start a simulation within framework of regularized code and integrate only one timestep. The output of this calculation in which we are interested in are the new acceleration values of one or two bodies moved by one step of the algorithm. The same input is used for Hermite code, so we start the simulation and carry it up to the time that we have integrated by the regularized code. After these actions, we can compare calculated accelerations, considering the Hermite code output as the true solution of the N-body problem. There is a number of ways to compare two codes that have been discussed in (Dehnen, 2014). We use the most trivial, plotting the distribution of errors normalized by acceleration calculated by Hermite integrator. This value is said to give the upper boundary to the real relative error, so our error estimations are somehow overstated. The Figure 2 represents this distribution:

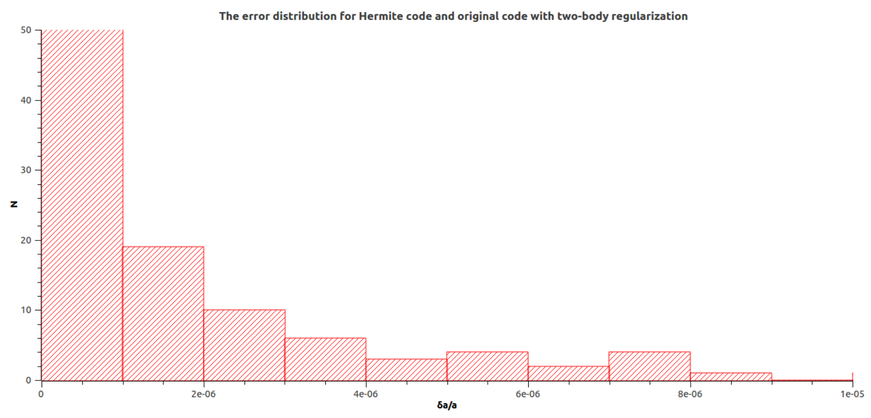

Figure 2: Error distribution histogram. The height of the left column exceeds 400.

The average error is on a level of 10^{-7}, which is in a good agreement with criteria for the collisional code (Dehnen, 2014).

Another useful experiment is carried out to distinguish the reduction of the timestep for the ordinary algorithm from the precision level that is allowed by two-body regularization. As is said above, we formulate the criteria to choose the integration either by Taylor expansion or with the use of the regularization. If we increase the parameter p, the possibility of the regularization to be turned on decreases incredibly, so in this limit we have the code with two layers of Ahmad-Cohen only. Comparison between this code and Hermite gives the average error of unregularized code is bigger than one achieved with regularization at least by the order of magnitude.

We also have conducted a numerical experiment analogous to one described in (Lezhnin, Chernyagin, 2014) in order to check the precision of integrals conservation. **The simulation on relaxation timescale gives the relative error up to 10^{-4}, which should be considered as a good result for nonsymplectic and nonsymmetric integration scheme.** The deviations of momentum of the system also appear to be small compared with its RMS value. The snapshot of this simulation is presented on the Figures 3. Red and purple mark the bodies from different clusters; turquoise marks bodies that are intergated with the use of regularization procedure.

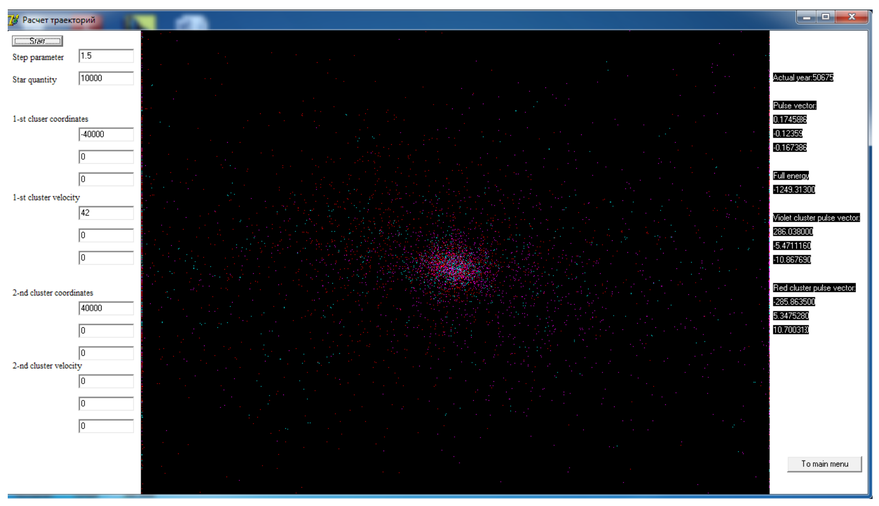

Figure 3: Figure 3. The snapshot of the evolution of the two star clusters, $t = t_{dyn}$

Discussion

We have discussed the final adjustments of the algorithm presented in (Lezhnin, Chernyagin, 2014). At the moment, it allows to conserve energy with the relative level of precision about 10^{-4}. The average error is in agreement with criteria for collisional code (Dehnen 2014). The further development of collisional N-body algorithm demands some schemes for efficient and accurate calculation of the far-field impact. The Ahmad-Cohen scheme for 100 nearest bodies, regularization (Lezhnin, Chernyagin, 2014) and individual timestep choice are combined within the framework of the original N-body code. However, it still has complexity $O(N^2)$, which is unacceptable for the galactic core and globular cluster simulation. In the article by Dehnen(2014), the fast multipole method is considered as the method for collisional modelling that allows to decrease the complexity at least to the level of $O(NlogN)$. Realization of this kind of algorithm alongside with regularization described above seems to be perspective instrument for collisional N-body modelling.

References

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